Research Statement

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1 Introduction

My research is primarily in the field of topology, specifically in low dimensional topology and knot theory. A knot K is an embedding of the unit circle S^1 into $S^3 := \mathbb{R} \cup \{\infty\}$. Two knots are equivalent if there is an ambient isotopy that takes one knot to the other. In general, there is no computationally viable way to tell if two knots are equivalent or not equivalent. So one employs various knot invariants, functions on knots that are invariant under ambient isotopy, to attempt to distinguish knots.

To give an intuitive idea of the main questions in knot theory, imagine an extension cord. Tie it up any way you can, then connect the ends by plugging it into itself. Is it possible to untie the extension cord without unplugging it? Usually someone will convince themselves that certain knots cannot be untangled, but in general, how can we be sure it cannot be untied? This is where invariants and diagrams come in. One such invariant, which is a personal favorite of mine is the genus of a knot, $g(K)$. This is the minimum genus of all embedded orientable surfaces in S^3 that have the knot K as the boundary. To be well-defined, we must first know that every knot bounds an orientable surface, and Seifert provided the algorithm that appears in most texts. Via the study of surfaces, we can better understand knots, which is an exciting marriage of knots and surfaces. See the figure 1.

This classic invariant is what sparked my interest in bridge surfaces, surfaces which are the key ingredient to construct the bridge spectra. Instead of the knot being the boundary of a surface, however, we construct surfaces without boundary that the knot intersects transversely. The surface can be thought of cutting through the knotted sections and what is left of the knot on each side of the surface are arcs which have no knotting. The genus g bridge number $b_g(K)$ is half of smallest number of times the knot needs to intersect a standardly embedded genus g surface while in this bridge position. The list of these numbers $(b_0(K), b_1(K), b_2(K), \ldots)$ is the bridge spectrum of K, and was defined in 2011 by Zupan [12]. Since it is such a new invariant, there is relatively little known about it. It is not hard to see that this list is strictly decreasing, and it was shown by Tomova that there are knots, called high distance knots, which have a bridge spectra that decreases by one with each step, so they attain the upper bound.

2 Background

In 1954, Schubert [10] defined the bridge number of a knot. There are numerous ways to define bridge number, but the one that our discussion best fits is the following. In a 3-manifold M, a properly embedded arc α is an embedding of the unit interval into M with $\partial\alpha \subset \partial M$. A properly embedded arc α is called trivial if there is an arc $\beta \subset \partial M$ such that $\alpha \cap \beta = \partial\alpha = \partial\beta$ and α and β bound a disk embedded in M. The condition of being trivial keeps an arc from being knotted. A classical result is that every 3-manifold M can be decomposed into $M = V_1 \sqcup_\Sigma V_2$, where V_i is a handlebody and $\partial V_i = \Sigma$ for $i = 1, 2$. If Σ is a genus g surface, this splitting is called a Heegaard decomposition of genus g. The 3-sphere S^3 has Heegaard splittings of every genus $g \geq 0$. Let J

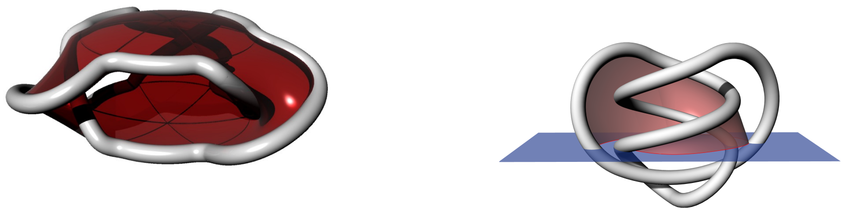

Figure 1: On the left, a knot with its accompanying Seifert surface. On the right, a knot in bridge position with a bridge disk shown.

be an embedded 1-manifold in M, then a bridge splitting of (M, J) is a Heegaard splitting of M such that J intersects Σ transversely, and $J \cap V_i$ is a collection of trivial arcs.

Definition 2.1 *For a knot $K \subset S^3$, the bridge number $b(K)$ is the minimum number of trivial arcs over any genus zero bridge splitting of (S^3, K).*

From here, one might ask why we only consider genus zero bridge splittings. This question is what spawned the genus g bridge number, and we suddenly have an invariant for every non-negative integer. This generalization was first studied by Doll in [3] and Morimoto and Sakuma in [8].

Definition 2.2 *For a knot $K \subset S^3$, the genus g bridge number $b_g(K)$ is the minimum number of trivial arcs over any genus g bridge splitting of (S^3, K).*

Then it is clear that the classical bridge number is the genus zero bridge number. We look at each of these numbers together, in what is known as the bridge spectrum of a knot, introduced by Zupan in [12].

Definition 2.3 *For a knot $K \subset S^3$, the bridge spectrum, $\mathbf{b}(K)$ is the list of genus g bridge numbers, where*

$$(b_0(K), b_1(K), b_2(K), \ldots)$$

and $b_g(K) = 0$ if K can be embedded into a genus g Heegaard surface.

Some use an alternate definition which may differs only in the last non-zero term.

Definition 2.4 *For a knot $K \subset S^3$, the primitive bridge spectrum, $\hat{\mathbf{b}}(K)$ is the list of genus g bridge numbers, where*

$$(b_0(K), b_1(K), b_2(K), \ldots)$$

and $b_g(K) = 0$ if K can be embedded into a genus g Heegaard surface and K is primitive on one side of the surface.

As the bridge spectrum is a relatively new invariant, there remain many interesting open questions, and many more questions that we have yet to ask. The bridge spectrum decreases by at least one, by a process called *meridional stabilization*. This means every spectra is bounded above by $(b_0(K), b_0(K) - 1, b_0(K) - 2, \ldots)$, and is eventually zero for all $g > b_0(K)$. A knot that attains these upper bounds is said to have a stair-step bridge spectrum. If $b_g(k) - b_{g+1}(K) = n > 1$, we say that K has a gap of size n at index g.

3 Results

As mentioned in the introduction, Tomova, in [11], found that a class of knots has the stair-step bridge spectra, called high distance knots. One may ask the following:

Question 3.1 *Is the class of high distance knots exactly the class of knots which attain the stair-step bridge spectra?*

In [6], I answer this question in the negative by proving the following two theorems. A generalized Montesinos knot is a knot obtained by a list of rational numbers $\beta_{i,j}/\alpha_{i,j}$, where $\{\alpha_{i,j}, \beta_{i,j} \in \mathbb{Z} : i = 1, \ldots, \ell, j = 1, \ldots, m\}$. Let $\alpha = \gcd\{\alpha_{i,j} : i = 1, \ldots, \ell, j = 1, \ldots, m\}$.

Theorem 3.2 *A generalized Montesinos knot or link, K with $\alpha \neq 1$ has the stair-step primitive bridge spectrum, $\hat{\mathbf{b}}(K) = (\hat{b}_0(K), \hat{b}_0(K) - 1, \ldots, 2, 1, 0)$.*

So I show that this class of knots also attains the upper bound for bridge spectra. But I also show that a subclass of these knots are not high distance.

Proposition 3.3 *If $K_n(p_1, \ldots, p_n)$ is a pretzel knot with $n \geq 4$, and P a genus zero bridge surface for K_n, then $d(P, K_n) = 1$.*

Combining these two theorems gives an answer question 3.1. My dissertation was focused on computing the bridge spectrum of cables of 2-bridge knots.

Theorem 3.4 *Let $K_{p/q}$ be a non-torus 2-bridge knot and $T_{m,n}$ an (m, n)-torus knot. If $K := \mathsf{cable}(T_{m,n}, K_{p/q})$ is a cable of $K_{p/q}$ by $T_{m,n}$, then the bridge spectrum of K is $\mathbf{b}(K) = (2m, m, 0)$.*

The proof of theorem 3.4 relies on Hatcher's and Thurston's paper [4] where they compute the boundary slopes of incompressible surfaces in 2-bridge knot complements. It also heavily relies on two results of Zupan [12], which allow us to to consider bridge surfaces in two disjoint cases. The one case is for strongly irreducible bridge surfaces and allows us to cut our bridge surface along properly embedded essential surfaces with the pieces of our bridge surface will be essential with possibly one strongly irreducible piece. Zupan's result has potential to aid future work of mine.

4 Future Work

4.1 Bridge spectrum

Hatcher and Oertel, in [5], extend the result of Hatcher and Thurston to Montesinos knots, which are a natural way to generalize 2-bridge knots. The bridge spectrum of a 2-bridge knot that is not a torus knot is $(2, 1, 0)$, and for a n branch Montesinos knot with $\alpha \neq 1$ has bridge spectra $(n, n-1, \ldots, 1, 0)$, it is reasonable to think that a cable of a Montesinos will exhibit the same behavior as a cable of a 2-bridge. Alas, this is not the case, we have examples where we find degenerate behavior in the bridge spectra.

Question 4.1 *What is the bridge spectrum of a cable of a Montesinos knot?*

I plan to investigate this question by applying techniques that are similar to what I used to compute the bridge spectra of cables of 2-bridge knots. Under the right conditions, we expect to find that some cables of Montesinos knots will have this m-stair-step, $(mn, m(n-1), \ldots, 2m, m, 0)$.

Zupan gave examples of knots which have arbitrarily large gaps and arbitrarily many of them in [12]. He then asks the following question:

Question 4.2 *What other spectra can be realized by knots in S^3? Specifically, for any decreasing sequence $\mathbf{v}$ of positive integers, is there a knot K such that $\mathbf{b}(K) = \mathbf{v}$?*

The scope of this question is very large, and thus is likely to remain open for some time. So finding interesting bridge spectra is our best way to begin to look for an answer. Twisted torus knots, which have been studied by Bowman, Taylor, and Zupan in [1], are another class of knots I intend to examine. The class of twisted torus knots are known to contain connected sums of torus knots, but in general it is unknown when they are hyperbolic.

But this leads to another inquiry, we are regularly making use of the structure of cabling to compute bridge spectra. Can we produce similar results without using these types of knots?

Question 4.3 *For any decreasing sequence $\mathbf{v}$ of positive integers, is there a hyperbolic knot K such that $\mathbf{b}(K) = \mathbf{v}$?*

More generally, in math we are always looking for links to other topics in math. We have already seen some relations between tunnel number and bridge spectra.

Question 4.4 *What connections are there between bridge spectrum and other invariants?*

4.2 Straight knots

Another area of research I am pursuing is an invariant of my own creation called the *straight number* of a knot, $\mathsf{str}(K)$. This invariant is defined on the diagram of a knot and is very well suited for undergraduate research. Every knot has a diagram which all the crossings occur on one *straight strand*, as shown in the figure below.

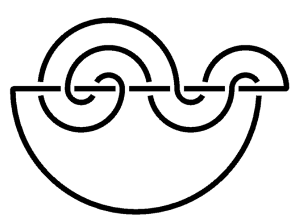

Figure 2: A knot in straight position.

This allows us to encode the information that we need to identify the knot, up to mirror images, with an element of the symmetric group on n elements, σ, decorated with crossing information, which we call the *straight code* of the knot, written $(\sigma(1)^{\pm}, \sigma(2)^{\pm}, \ldots, \sigma(n)^{\pm})$. But this gives us an interesting fact about diagrams: Every knot has a diagram which is made of up exactly one straight strand and some number of semi-circles with their centers on the straight strand.

A very valuable fact about this invariant is that there are only a finite number of straight codes of length n. Clearly, there is an upper bound of $n!2^n$ straight codes, for there are $n!$ permutations and each has 2^n choices of over or under for each crossing. There are many ways lower this bound but it does grow fast regardless. One method for tackling this problem has been a exhaustive check on each possibility. I have developed some python code that I use along with SnapPy [2] to check these large lists of knots. SnapPy is a program that was developed to look into the topology of 3-manifolds. Currently I am working on computing the invariant for the entire standard table of knots found in Rolfsen, [9] and many others.

A common question for knot invariants is how it behaves under connected sum, combining two knots, which we write $K\#J$ for knots K and J. We conjecture the following:

Conjecture 4.5 *Given knots K and J,* $\mathsf{str}(K\#J) = \mathsf{str}(K) + \mathsf{str}(J)$.

We are currently using ideas similar to Menasco, [7] to investigate this conjecture.

Looking into the straight number of different classes of knots is a focus of this research also. One particular class is again 2-bridge knots because of their inherent structure. A 2-bridge knot has a rational number associated to it and a partial fraction decomposition. The number theory that comes along with this fact could play an integral role in answering the following question.

Question 4.6 *What is the straight number of a 2-bridge knot?*

4.3 3D modeling and printing

The interest in 3D printing has been growing for some time now. There are many questions that are arising as a result of more and more people start to use this technology and very few mathematical papers have been written on the subject. There are two main areas I would like to investigate: how to produce high quality models and the physical process of the 3D printing. Both offer a variety of questions and here are a few.

Question 4.7 *What topics in mathematics would benefit from a 3D model?*

This is a large and broad questions that does not have a specific answer, but one that we should keep on our minds, in case we stumble across something. The next question deals with .stl files, which are what most 3D printers read in to create a print.

Question 4.8 *Can we expedite the process of creating .stl files without the use of extra software?*

This is a question which Kyle Istvan and I are trying to answer for Riemann Sums of surfaces, which I mention in my teaching statement. We will post the program we build online for anyone to use, so they can create their own .stl files quickly and easily.

References

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