Research Statement

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1 Introduction

My research is primarily in the field of topology and geometry, specifically in low dimensional topology and knot theory. A knot K is an embedding of the unit circle S^1 into $S^3 := \mathbb{R}^3 \cup \{\infty\}$. Two knots are equivalent if there is an ambient isotopy that takes one knot to the other. In general, there is no computationally viable way to tell if two knots are equivalent or not. So one employs various knot invariants, functions on knots that are invariant under ambient isotopy, to attempt to distinguish knots.

To give an intuitive idea of the main questions in knot theory, imagine an extension cord. Tie it up any way you can, then connect the ends by plugging it into itself. Is it possible to untie the extension cord without unplugging it? Usually someone will convince themselves that certain knots cannot be untangled, but in general, how can we be sure when it cannot be untied? This is where invariants and diagrams come in. A knot diagram is a projection of the knot to a 2-dimensional plane with a finite number of double points and no points of higher intersection. At each point of intersection, we identify which strand is "over" the other. From these diagrams, we want to be able to say something about the knot. An amazing result of Menasco [10] tells us that if the knot diagram is alternating and reduced, then it cannot be simplified further. And therefore it cannot be untied! This result takes the combinatorics of a diagram and tells us topological properties of the knot.

So, from Menasco, the diagram alone is enough to get information about the knot. What other results can we deduce from the diagram alone? In the 1980's, W. Thurston showed that there are three types of knots, the most common being hyperbolic knots. A knot is hyperbolic if its complement in S^3 admits a geometric structure with a complete hyperbolic metric. This shows that the topology of knots and their geometry are intrinsically related. The results that followed started to give us ways to estimate the hyperbolic volume of a knot, another important invariant, just from the diagram. For instance, D. Thurston shows that for any link K with a diagram that has n crossings,

$$\mathrm{vol}(K) \leq v_{oct} n$$

where $v_{oct} \approx 3.6638$ is the volume of the ideal hyperbolic octahedron. Thus, we can merely look at the combinatorial properties of a diagram and obtain geometric results. I consider these results to be some of the most interesting in knot theory. My current work is in the vein of these ideas: given combinatorial information about a diagram, what topological or geometric properties are present?

By using a result of Adams, Shinjo, and Tanaka, we know that every knot has a diagram with all crossings on a single horizontal arc of the knot, and we say a diagram like this is a *straight diagram*, see Figure 1. To obtain a straight diagram from a diagram with the minimum number of crossings, we likely had to increase the number of crossings. Thus, we make an invariant called the *straight number* of a knot K, $\texttt{str}(K)$, which is the minimum number of crossings of any straight diagram of K.

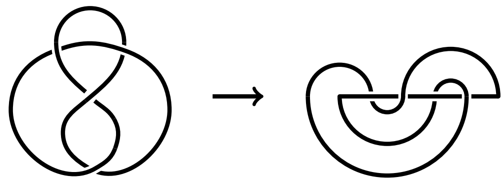

Figure 1: A diagram of a knot in straight position and its simplified diagram.

With this structure, what combinatorial properties are present which allow us to say something about the topology and geometry of the knot? This is, in general, an open question. Here we present some new results in this direction using the structure of straight knots.

2 Straight number

We make use of a theorem by Adams, Shinjo, and Tanka to begin.

Theorem 2.1 *[1, Theorem 1.2] Every knot has a projection that can be decomposed into two sub-arcs such that each sub-arc never crosses itself.*

From this theorem we know that every knot can be drawn with two arcs where all crossings occur between the these two arcs. By planar isotopy, we can make one of these arcs straight, and we say the diagram is in *straight position.*

Definition 2.2 *Given a knot K, the* straight number *of K, $\texttt{str}(K)$, is the minimum number of crossings over all diagrams of K that are in straight position.*

If a knot K has $\texttt{str}(K) = c(K)$, where $c(K)$ is the crossing number, then we say K is *perfectly straight.* The horizontal arc through the middle of the diagram is called the *straight strand.*

One can check the following theorem.

Theorem 2.3 (O., 2018)

1. *Every torus knot $T_{2,q}$ is perfectly straight.*
2. *Every n-pretzel knot is perfectly straight.*
3. *Every 2-bridge knot $K_{p/q}$ where the continued fraction of p/q has length less than 6 is perfectly straight.*

This next theorem was proved with the use of many Python scripts that I developed and wrote to efficiently build a list of all possible straight knots, the topology and hyperbolic geometry program SnapPy, [4] and a super computer. To achieve this, I employed the fact that there are a finite number of knots with straight number n. And it is possible to enumerate all diagrams in straight position with n crossings. Thus, we can just look at the list we create and identify all these knots. Naturally, many of these diagrams will be of knots with straight number less than n. So we proceed from $n = 3$ and increase n, identifying each diagram. This process allowed us to calculate the straight number for the standard table of knots up to 10 crossings in Rolfsen [19]. Here are the first examples of knots which are not perfectly straight.

Theorem 2.4 (O. 2018) *Let K be a knot with 10 or less crossings. Then K is perfectly straight or $\texttt{str}(K)$ is given below.*

$$
\begin{array}{ll}
\texttt{str}(8_\alpha) = \texttt{str}(9_\beta) = 10, & \alpha \in \{16, 18\}, \beta \in \{32, 47\} \\
\texttt{str}(9_\gamma) = \texttt{str}(10_\delta) = 11, & \gamma \in \{29, 33, 34, 41\}, \delta \in \{69, 75, 97, 101, 165\} \\
\texttt{str}(9_{40}) = \texttt{str}(10_\epsilon) = 12, & \epsilon \in [84, 89] \cup [91, 93] \cup \{96, 100\} \cup [103, 105] \cup \\
& \qquad \{108\} \cup [111, 114] \cup [116, 119] \cup \{122, 123\} \\
\texttt{str}(10_\zeta) = 13, & \zeta \in \{102, 121\}
\end{array}
$$

Note that of the 250 knots with 10 or less crossings, only 41 of them are not perfectly straight and appear in the list above. One can ask what families of knots are not perfectly straight and what is the largest straight number can be given the crossing number of a knot? We begin to answer these questions with the following theorems.

Theorem 2.5 (O. 2018) *Given a knot K, $\texttt{str}(K) \leq 2^{c(K)-1} - 1$.*

The proof of this theorem comes from a careful counting argument of the algorithm described by Adams, Shinjo, and Tanaka. But it begs the question, is there a family of knots which attain this upper bound, i.e., how sharp is this bound? We guess it is not sharp, but there may be a family which does require an exponential increase in the number of crossings.

As to what knots might be candidates for the largest differences in straight number and crossing number, we look to weaving knots. A weaving knot, $W(p, q)$, is obtained by making a torus knot $T_{p,q}$ alternating.

Theorem 2.6 (O. 2018) *Let $n \geq 3, m \geq n+1$ and $\gcd(n, m) = 1$. Every Weaving knot $W(n, m)$ is not perfectly straight, i.e.*

$$c(W(n, m)) < \texttt{str}(W(n, m)).$$

The proof of this result uses the classic results of Menasco and Thistlethwaite that the Tait flyping conjecture is true, [11]. In addition, we then need Menasco's result "an alternating knot is prime if and only if it looks prime," [10]. With these two important results about the combinatorics of alternating diagrams, we again perform a careful counting argument.

Given a (not) perfectly straight alternating knot, K, the next theorem allows us to create families of alternating knots $\{K_\alpha\}$ that are also (not) perfectly straight. A *twist region* is a maximal sequence of bigons and *increasing the number of full twist* means adding in an even number of bigons to that region.

Theorem 2.7 (O. 2018) *Let K be an alternating knot. Given any minimal diagram D of K, let K' be the knot obtained by increasing the number of full twists in any twist region of D. Then K is perfectly straight if and only if K' is perfectly straight.*

This lets us add full twists to our weaving knots and does not change the conclusion of theorem 2.6. More importantly, this theorem allows us to build the first infinite family of not perfectly straight knots for which we explicitly know the straight number.

Theorem 2.8 (O. 2018) *Let $t = (t_1, t_2, \ldots, t_6)$ be positive integers such that t_1, t_2, t_5, and t_6 are odd and t_3 and t_4 are even and let s be the sum of the t_i's. Let K_t be the alternating knot obtained from the template in Figure 2 with t_i crossings in the corresponding twist region. Then $\texttt{str}(K_t) = c(K_t) + 1 = s + 2$.*

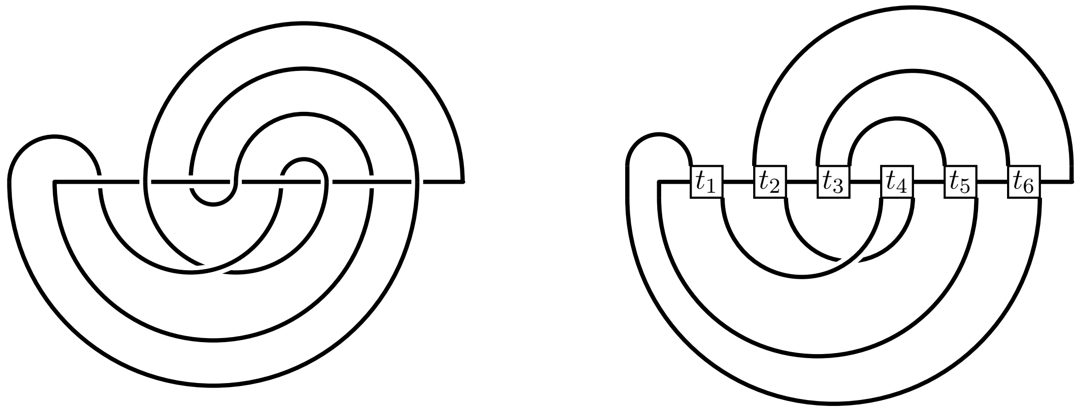

Figure 2: On the left, the knot 9_{32}. On the right, the template for Theorem 2.8. It is made by adding full twists to every twist region of 9_{32} except the one crossing not on the straight strand.

3 Bridge Spectrum

This section gives a very brief summary of the main result of my thesis, [13]. In 1954, Schubert [20] defined the bridge number of a knot. There are numerous ways to define bridge number, but the one that our discussion best fits is the following. In a 3-manifold M, a properly embedded arc α is an embedding of the unit interval into M with $\partial\alpha \subset \partial M$. A properly embedded arc α is called trivial if there is an arc $\beta \subset \partial M$ such that $\alpha \cap \beta = \partial\alpha = \partial\beta$ and α and β bound a disk embedded in M. The condition of being trivial keeps an arc from being knotted. A classical result is that every 3-manifold M can be decomposed into $M = V_1 \sqcup_\Sigma V_2$, where V_i is a handlebody and $\partial V_i = \Sigma$ for $i = 1, 2$. If Σ is a genus g surface, this splitting is called a Heegaard decomposition of genus g. The 3-sphere S^3 has Heegaard splittings of every genus $g \geq 0$. Let J be an embedded 1-manifold in M, then a bridge splitting of (M, J) is a Heegaard splitting of M such that J intersects Σ transversely, and $J \cap V_i$ is a collection of trivial arcs.

Definition 3.1 *For a knot $K \subset S^3$, the bridge number $b(K)$ is the minimum number of trivial arcs over any genus zero bridge splitting of (S^3, K).*

From here, one might ask why we only consider genus zero bridge splittings. This question is what spawned the genus g bridge number, and we suddenly have an invariant for every non-negative integer. This generalization was first studied by Doll in [5] and Morimoto and Sakuma in [12].

Definition 3.2 *For a knot $K \subset S^3$, the genus g bridge number $b_g(K)$ is the minimum number of trivial arcs over any genus g bridge splitting of (S^3, K).*

Note that the classical bridge number is the genus zero bridge number. We look at each of these numbers together, in what is known as the bridge spectrum of a knot, introduced by Zupan in [21].

Definition 3.3 *For a knot $K \subset S^3$, the bridge spectrum, $\mathbf{b}(K)$ is the list of genus g bridge numbers, where*

$$(b_0(K), b_1(K), b_2(K), \ldots)$$

and $b_g(K) = 0$ if K can be embedded into a genus g Heegaard surface.

The bridge spectrum necessarily decreases by at least one, through a process called *meridional stabilization*. This means every spectra is bounded above by $(b_0(K), b_0(K)-1, b_0(K)-2, \ldots)$, and is eventually zero for all $g > b_0(K)$. A knot that attains this upper bound is said to have a stair-step bridge spectrum. If $b_g(k) - b_{g+1}(K) = n > 1$, we say that K has a gap of size n at index g. My dissertation was focused on computing the bridge spectrum of cables of 2-bridge knots.

Theorem 3.4 (O. 2016) *Let $K_{p/q}$ be a non-torus 2-bridge knot and $T_{m,n}$ an (m,n)-torus knot. If $K := \mathsf{cable}(T_{m,n}, K_{p/q})$ is a cable of $K_{p/q}$ by $T_{m,n}$, then the bridge spectrum of K is $\mathbf{b}(K) = (2m, m, 0)$.*

The proof of theorem 3.4 is broken into two main cases using the famous result of Casson and Gordan that a splitting is either strongly irreducible or weakly reducible. In both cases, theorems of Zupan [21] are then applied to give us more control over the surfaces. Then we make use of classical work of Hatcher's and Thurston's paper [8] where they compute the boundary slopes of incompressible surfaces in 2-bridge knot complements, as well as many others.

4 Future Work

4.1 Straight knots

In the work to prove Theorem 2.4, we noticed that there seemed to be a relation between straight number and volume. For fixed crossing number, the knots which had higher volumes are more likely to be the knots which also have higher straight numbers. This general idea has spawned two different directions of investigation. The first, is to create a random knot model and with it, look at the expected value of the volume of a random knot. This work is joint with Anastasiia Tsvietkova, my unit lead at OIST. The second is to look at volumes of a special class of knots called *Snail knots*. These knots are perfectly straight and are candidates for the highest volume knots with n crossings.

4.1.1 Random Straight Links

Above we state that every knot has a straight diagram. This does not hold for every link though. We will create a random model that contains all knots and some links, then expand it to contain all links. Let S_{top}, S_{bot} be two random valid strings of s pairs of parenthesis. We let the matched parentheses define the endpoints of semicircles, and when lined up as in Figure 3, we obtain a random link.

For example when $s = 5$, we might have $S_{top} = (\,(\,)\,(\,)\,)\,(\,(\,)\,)$ and $S_{bottom} = (\,)\,(\,(\,(\,)\,(\,)\,)\,)$, yielding the following link projection.

If we then allow for r parallel copies of each component, and use a new result of Even-Zohar, Hass, Linial, and Nowik, [7], we can now obtain all links. We call these (s,r)-random straight links. What can we show with this random model? The first thing one usually wants to know about a random model is if it is likely to produce nontrivial links.

Theorem 4.1 *Let L be a (s,r)-random link. If L is alternating and s tends to infinity, then L is nontrivial with probability one. If L has its crossings chosen at random and s and r tend to infinity, then L is nontrivial with probability one.*

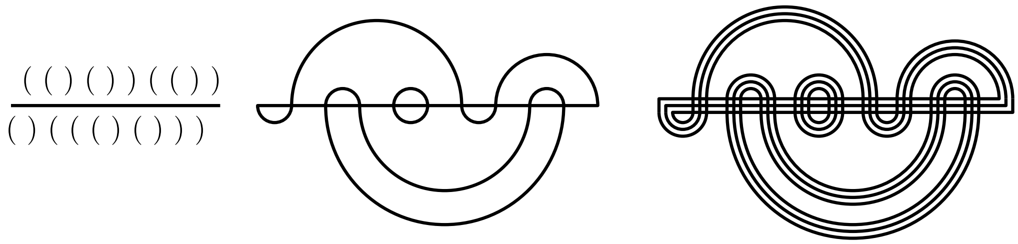

Figure 3: On the left, a pair of strings of parentheses, in the middle, the (5,1)-random straight link, and on the right, the corresponding (5,3)-random straight link projection.

Surprisingly, to show Theorem 4.1, we use a classic result of Poincaré [17], that is usually applied to differential equations and an algorithm called Zeilbergers algorithm which finds recursive relations [16].

Next, we use the classic results from Lackenby, Agol and Thurston.

Theorem 4.2 ([9]) *Let D be a prime alternating diagram of a hyperbolic link L in S^3. Then*

$$v_3(t(D)-2)/2 \leq Volume(S^3 - L) < 10v_3(t(D)-1).,$$

where $v_3(\approx 1.01494)$ is the volume of a regular hyperbolic ideal 3-simplex.

The upperbound is actually an improvement on Lackenby's original upperbound by Agol and Thurston. Also, the upperbound does not require the link be prime and alternating.

By applying this theorem to our random straight links, we arrive at the following expected volume of the complement.

Theorem 4.3 (O., Tsvietkova 2018) *Let D be a prime alternating diagram of a hyperbolic link L in S^3 obtained from a (s, r)-random straight link. Then*

$$v_3((2s-1)r^2 - s - 3)/2 \leq \mathbb{E}(Volume(S^3 - L)) \leq 10v_3((2s-1)r^2 - s - 3),$$

where $v_3(\approx 1.01494)$ is the volume of a regular hyperbolic ideal 3-simplex. In particular, when $r = 1$, L has $2s - 1$ crossings and

$$v_3(s-4)/2 \leq \mathbb{E}(Volume(S^3 - L)) \leq 10v_3(s-3).$$

Again, the upperbounds here do not require the link be prime and alternating.

Our next step is improve these bounds by applying the work of Dashbach and Tsvietkova [6]. This involves analysis of the number of twist regions and how many bigons are present in them. We also hope to find other invariants for which we can bound the expected value.

4.1.2 Snail Links

Champanerkar, Kofman, and Purcell note in their paper [3] that Xiao-Song Lin conjectured that Weaving links are the highest volume knots for fixed crossing number. In recent work with Jessica Purcell, we have numerical data which suggests that there is a class of knots which has higher volume. These knots are called *snail links* and were discovered in the process of proving Theorem 2.4.

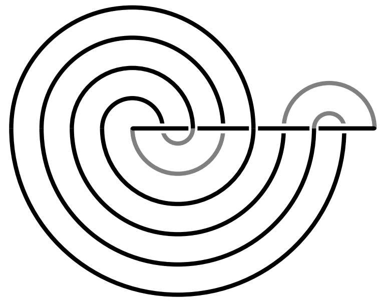

Figure 4: The snail link $\mathsf{snail}(2,7)$. The two semicircles defined by $s = 2$ on each end of the straight strand are colored gray. There is a unique way to add in the rest of the black arcs.

A *snail link*, $\mathsf{snail}(s,c)$ is an alternating link with s semicircles on opposite sides and ends of the straight strand and c crossings, with the condition that $c \geq 2s-1$. See Figure 4.

This family of links has many interesting properties. We know that $\mathsf{snail}(2,c) = \mathsf{snail}(\frac{c+a}{3}, c)$ where $a = 2$ or 3, and moreover, these are 2-bridge links. When $c = 2s-1$, these are a different 2-bridge knot. It is our conjecture that these are the only cases when a snail link is a 2-bridge link. Looking at numerical data for the first 250,000 snail links, we have the following conjecture and question.

Conjecture 4.4 *For fixed c, the three 2-bridge snail links above, when they exist, are the smallest volume snail links.*

Question 4.5 (O., Purcell 2018) *For fixed c, is the largest volume snail link is the largest volume link with c crossings?*

4.1.3 Straight number under knot operations

A common question for knot invariants is how it behaves under connected sum, combining two knots, which we write $K\#J$ for knots K and J. We conjecture the following:

Conjecture 4.6 *Given knots K and J, $\mathsf{str}(K\#J) = \mathsf{str}(K) + \mathsf{str}(J)$.*

We are currently using ideas similar to Menasco, [10] to investigate this conjecture. Next, we guess that cabling, another operation, will drastically increase the straight number.

Question 4.7 *How does straight number behave under cabling?*

4.2 Bridge spectrum

Given a knot K, can we compute the bridge spectrum? In general, this is a very difficult problem, see Bowman, Taylor, Zupan [2]. There is not even a computationally reasonable way to compute $b_0(K)$, the very first number in the list. Perhaps a more tenable question would be the following.

Question 4.8 *For any decreasing sequence $\mathbf{v}$ of positive integers, is there a knot K such that $\mathbf{b}(K) = \mathbf{v}$?*

More generally, in math we are always looking for links to other topics in math. In my thesis, some relations between tunnel number and bridge spectrum are investigated. A recent paper of Purcell and Zupan, [18] shows that hyperbolic volume and the genus g bridge number are independent.

Question 4.9 *What connections are there between bridge spectrum and other invariants?*

4.3 3D modeling and printing

The interest in 3D printing has been growing for some time now. There are many questions that are arising as a result of more and more people start to use this technology and very few mathematical papers have been written on the subject. There are two main areas I would like to investigate: how to produce high quality models and the physical process of the 3D printing. Both offer a variety of questions and here are a few.

Question 4.10 *What topics in mathematics would benefit from a 3D model?*

This is a large and broad questions that does not have a specific answer, but one that we should keep on our minds, in case we stumble across something. The next question deals with .stl files, which are what most 3D printers read in to create a print.

Question 4.11 *Can we expedite the process of creating .stl files without the use of extra software?*

This is a question I am trying to answer for Riemann Sums of surfaces, which I mention in my teaching statement. We will post the program we build online for anyone to use, so they can create their own .stl files quickly and easily.

5 Undergraduate and Graduate Research

Here, I want to briefly mention some of the possible avenues of research that I hope to advise students down. Knot theory has many important open questions that can be understood without lots of background knowledge. Diagrams and the combinatorial aspects of knots are often accessible to undergraduates. For example, the following famous conjecture is open.

Conjecture 5.1 *Given two knots K and J, $c(K\#J) = c(K) + c(J)$, where $c(K)$ is the crossing number of K.*

I remember when I first started research in knots and being able to completely understand current open questions was very exciting for me. I hope to give students the same excitement of discovering new questions for them to work on. Specifically, straight number has open questions which are appropriate for undergraduate research. It also has questions which are more appropriate for graduate research. For example, an undergraduate might find more families of knots which are not perfectly straight. A strong undergraduate or graduate student could look into the relation between straight number and other invariants. Knowing that straight number is independent from an invariant is just as important as knowing it is dependent. There are many options and sharing my knowledge and seeing a student discover their own results is something I truly look forward to.

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