Bridge spectra of cables of 2-bridge knots

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Outline

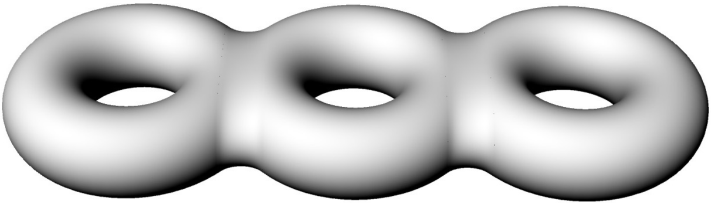
The outline for this talk

1. Bridge spectra and other definitions
2. Basic examples and results
3. Cables of 2-bridge knots
4. Questions

Background

A *Heegaard surface* is from a decomposition of a 3-manifold into handlebodies.

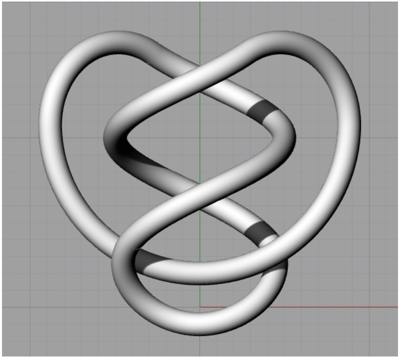
If that is not something you are used to, just this:



Background

Bridge number is an invariant usually defined from the diagram.

The *bridge number* of a knot is the minimum number of local maximums over all diagrams of the knot.

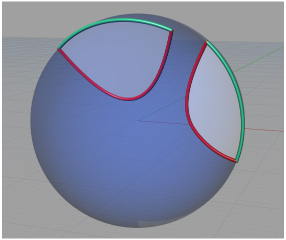


Background

But we want to think about this more geometrically. To do this, we will use another definition of bridge number.

A *trivial arc* α in B^3, with $\partial B^3 = S^2$, is a properly embedded arc that cobounds a disk, $D \subset B^3$ with an arc $\beta \in S^2$.

We call D a *bridge disk*.



Background

Let Σ be a Heegaard sphere that intersects the knot K transversely and cuts K into a collection of trivial arcs on both sides of Σ.

Notice that there will be the same number of trivial arcs on both sides.

Given a knot K in S^3, we can define the bridge number of K to be the minimum number of trivial arcs one side of Σ, for any bridge splitting sphere, Σ.

Now we will look at an example.

Generalizing bridge number

Fact: For every non-negative integer g, there genus g Heegaard splitting of $S^3 = V_1 \cup_\Sigma V_2$, where $g(\Sigma) = g$.

So, why did we only use genus zero Heegaard splittings?

Definition

A (g, b)*-splitting* for a knot K is a Heegaard splitting of S^3, such that Σ is a genus g surface and transverse to K and $V_i \cap K$ is a collection of b trivial arcs for $i = 1, 2$.

And then:

Definition (Morimoto and Sakuma, 1991 and Doll, 1992)

The genus g bridge number, $b_g(K)$, is the minimum b such that there exists a (g, b)-splitting of K

Bridge Spectrum

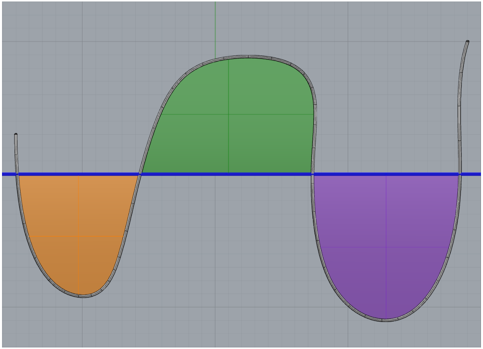
And now, we finally have what we need to define the invariant this talk is about.

Definition (Zupan, 2013)

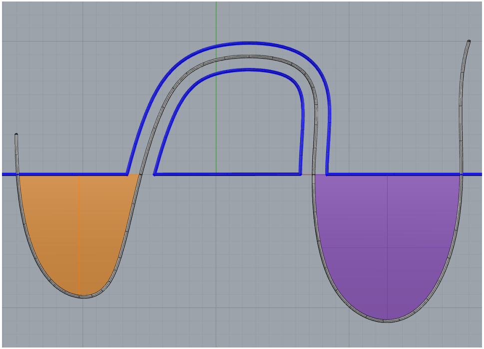
The *bridge spectrum* of a knot, $\mathbf{b}(K)$, is the list of genus g bridge numbers,

$$(b_0(K), b_1(K), b_2(K) \ldots)$$

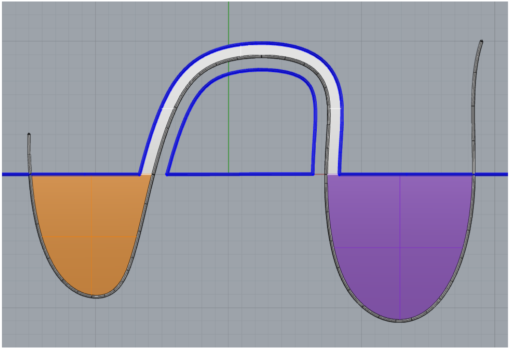
Meridional stabilization



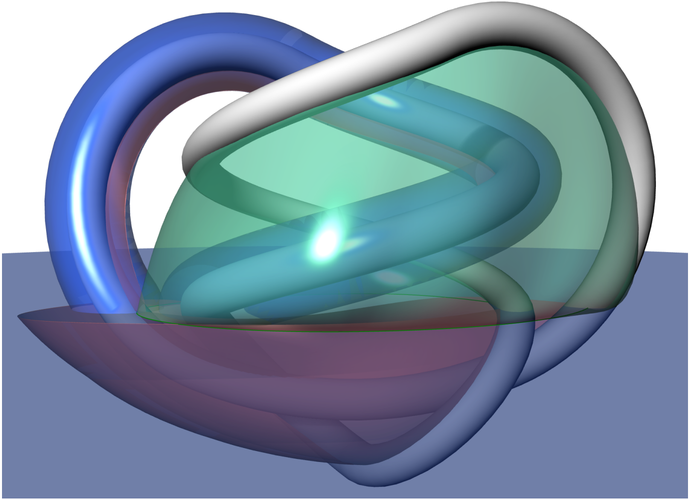
Meridional stabilization



Meridional stabilization



Meridional stabilization



Meridional stabilization

What this tells us:

Given a (g, b)-splitting with $b \geq 1$, we can form a $(g+1, b-1)$-splitting, *meridional stabilization*.

Bridge spectra are necessarily strictly decreasing sequences and the bridge spectrum for any knot K is bounded above by the sequence

$$(b_0(K), b_0(K) - 1, b_0(K) - 2, \ldots).$$

We call these sequences the *stair-step* bridge spectra.

Examples

Torus knots:

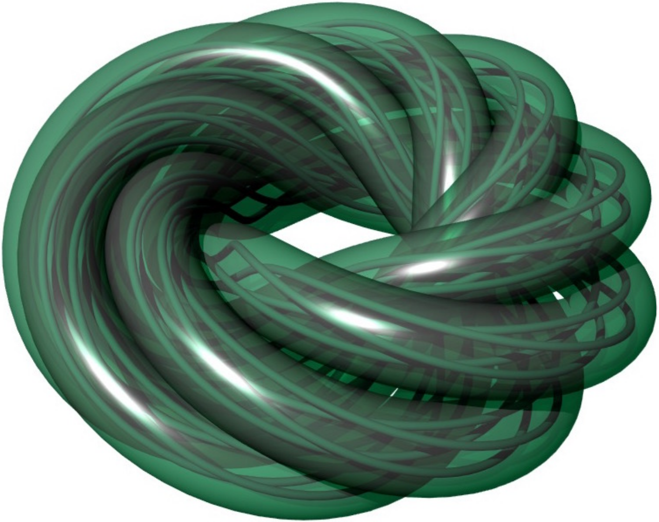
$\mathbf{b}(T_{p,q}) = (\min\{p,q\}, 0)$

$b_1(T_{p,q}) = 0$ as torus knots embed into the surface.

2-bridge knots

$\mathbf{b}(K_{m/n}) = (2,1)$ if it is not a torus knot.

Iterated torus knots



Iterated torus knots

Theorem (Zupan, 2013)

Suppose that $K_n = ((p_0, q_0), \ldots, (p_n, q_n))$ *is an iterated torus knot, whose cabling parameters satisfy* $|p_i - p_{i-1}q_{i-1}q_i| > 1$. *Then*

$$b_g(K_n) = \begin{cases} q_n \cdot b_g(K_{n-1}) & \text{if } g < n \\ \min\{|p_n - p_{n-1}q_{n-1}q_n|, q_n\} & \text{if } g = n \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$\mathbf{b}(K_n) = q_n \cdot \mathbf{b}(K_{n-1}) + \min\{|p_n - p_{n-1}q_{n-1}q_n|, q_n\} \cdot e_n.$$

Bridge number of a satellite knot:

Theorem (Schubert 1954, Schultens 2001)

Suppose K is a satellite knot with companion J, companion torus V, pattern (V, L) and index k. Then $b_0(K) \geq k \cdot b_0(J)$.

What this means: Schubert's result does not generalize to higher genus bridge number.

We see degenerate behavior of bridge spectra under the satellite operation.

Iterated torus knots

First, we will see the genus zero splitting, where Schubert's result does hold.

Then we will look at why it doesn't hold for higher genus.

Consider the cable knot, $K = T_{4,21} \hookrightarrow T_{2,3}$.

Pictures!

So, $\mathbf{b}(K) = (8, 3, 0)$ since we could do better than $(8, 4, 0)$.

Cables of 2-bridge knots

Theorem (O., 2016)

Let $K_{p/q}$ be a non-torus 2-bridge knot and $T_{m,n}$ an (m,n)-torus knot. If $K := T_{m,n} \hookrightarrow K_{p/q}$ is a cable of $K_{p/q}$ by $T_{m,n}$, then the bridge spectrum of K is $\mathbf{b}(K) = (2m, m, 0)$

We call this m-stair step bridge spectra.

Proof idea

We want to compute the b_0, b_1, and b_2.

Easiest first:

$b_0(K) = 2m$ by Schubert's result.

Next easiest:

$b_2(K) = 0$ because it embeds on a genus 2 surface.

Proof idea

Last, we want to find $b_1(K)$ and this is 95% of the work.

And its just for the lower bound, because I just showed you a picture where we get $b_1(K) \leq m$.

Some of the ingredients of the proof:

- A. Hatcher and W. Thurston, *Incompressible surfaces in 2-bridge knot complements*
- K. Morimoto, M. Sakuma, and Y. Yokota, *Identifying tunnel number one knots*
- A. Zupan, *Bridge spectra of iterated torus knots*
- C. McA. Gordon and R. A. Litherland,*Incompressible planar surfaces in 3-manifolds*

Extending the result

A natural way to generalize 2-bridge knots is a class of knots called Montesinos knots.

Theorem (O., 2015)

A generalized Montesinos knot or link, K with $\alpha \neq 1$ has the stair-step primitive bridge spectrum,
$\hat{\mathbf{b}}(K) = (\hat{b}_0(K), \hat{b}_0(K) - 1, \dots, 2, 1, 0)$.

The bridge spectrum of a Montesinos knot (with some conditions) is stair-step.

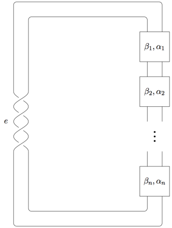
So one might think that we could expect to get m-stair-step.

One would be wrong with this assumption.

We have already observed cases where we get degeneration of bridge number of cables of pretzel knots, perhaps similar to the kind found in Zupan's paper.

Example

Here is the Montesinos link $M(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \ldots, \frac{\beta_n}{\alpha_n}|e)$



If $\alpha = \gcd\{\alpha_i\} \neq 1$, then this knot is stair step

Questions

Question

What other spectra can be realized by knots in S^3? Specifically, for any decreasing sequence $\mathbf{v}$ of positive integers, is there a knot K such that $\mathbf{b}(K) = \mathbf{v}$?

Question

What is the bridge spectrum of a cable of a Montesinos knot?

Question

For any decreasing sequence $\mathbf{v}$ of positive integers, is there a hyperbolic knot K such that $\mathbf{b}(K) = \mathbf{v}$?

Thanks

Thank you!