MA 121

Section 4.1 Worksheet March 30, 2020

Directions: It is suggested that you try actually reading the book sections before you watch the videos and attempt to fill out this worksheet. Please read the following problems and explanations carefully. Try to produce full, clear solutions to the exercises. These will help further your understanding.

Section 4.1: We start to apply our knowledge of derivatives to say exactly where the graphs change.

Recall, a function $f(x)$ is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. So if a function is going to change from increasing to decreasing, or vice versa, then $f'(x) = 0$ or where f' is undefined. And these x values are so important, we name them **critical points**.

At these critical points, there is a horizontal tangent line. If $f(x)$ changes from increasing to decreasing at this critical point, then there is a local maximum there. And if it changes from decreasing to increasing, it is a local minimum.

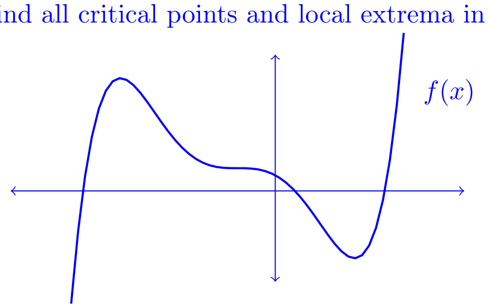
BUT!! this does not mean that every critical point is a max or min!

Practice 1:

(1) Sketch a graph with two local maxima and one local minima. Make to sure identify the critical points.

(2) Sketch a graph with a critical point but NO extrema at all.

(3) Find all critical points and local extrema in the following graph and label them.



Now we can think about how we would find these features of the function without seeing the graph. Let's do a simple example first.

Example W1 Let $f(x) = x^3 - 2x$. Where is f increasing and decreasing? Where are the local extrema, if any?

Solution Okay, to do this problem, we first start by find the critical points. To do this, we will need to find when $f'(x) = 0$. (This will almost always be the first step!) So let's take the derivative first, and we can use all the shortcuts we want.

$$f'(x) = 3x^2 - 2$$

Now we will find when $f'(x) = 0$. Because this is where the tangent line is horizontal, and if there is a minima or maxima, it will have a flat tangent line! So we solve the following equation for x.

$$0 = 3x^2 - 2$$

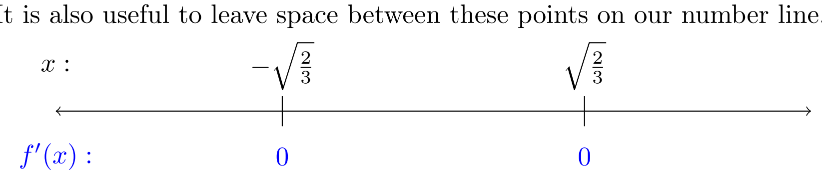
$$\frac{2}{3} = x^2$$
$$\pm\sqrt{\frac{2}{3}} = x$$

Right! Don't forget, when we take a square root, we need a plus or minus! So there are two critical points here, $x = \sqrt{\frac{2}{3}}$ and $x = -\sqrt{\frac{2}{3}}$. Now, the question is, are these critical points maxima or minima? We use the First Derivative Test.

To answer this question, we make a number line with our two critical points on it. (I will be redrawing the number line, so you can see the train of thought. You only need to draw one when you are doing these problems.)

These two x values are the critical points. We found them by solving the equation when $f'(x) = 0$, as you remember.

So, we place a row below the number line with the correspond values of – THIS IS IMPORTANT – $f'(x)$. The original function, $f(x)$ is not as important right now. (Note: New additions to our number line will be in blue.) It is also useful to leave space between these points on our number line.



Now here is the valuable part: If the derivative is positive, the function is increasing. We already knew this, but now we will use it explicitly. Since we know every point where the function has critical points, it is either increasing or decreasing at all the spots between the critical points. For this example, there are three intervals. These are $\left(-\infty, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$, and $\left(\sqrt{\frac{2}{3}}, \infty\right)$.

So in these intervals, the function is going up or going down. For the whole interval. So to find out which, all we need to do is check ONE – count 'em – point! And when I say check, I mean, plug these new picked points into f'. If f' is positive, the function is increasing on the whole interval. Negative, then its decreasing.

And we get to pick the points to check. So pick easy points! Like zero, and ones and other whole numbers, or numbers that will work out nicely.

For us, lets pick 0 in $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$, -2 in $\left(-\infty, -\sqrt{\frac{2}{3}}\right)$, and last, pick 1 in $\left(\sqrt{\frac{2}{3}}, \infty\right)$. Then we plug these into $f'(x) = 3x^2 - 2$. This number line is not to scale, and that really doesn't matter here. All that matters is that we pick any number IN the interval. Here, $\sqrt{\frac{2}{3}} \approx .816$, so 1 is in the interval $\left(\sqrt{\frac{2}{3}}, \infty\right)$.

$x:$	-2	$-\sqrt{\frac{2}{3}}$	0	$\sqrt{\frac{2}{3}}$	1
$f'(x):$	10	0	-2	0	1

Now, the values we got out of $f'(x)$ are exact here, but all we really care about is whether they are positive or negative. So you don't have to calculate them all out. You might prefer instead to do this:

$x:$	-2	$-\sqrt{\frac{2}{3}}$	0	$\sqrt{\frac{2}{3}}$	1
$f'(x):$	$+$	0	$-$	0	$+$

And to really drive the point home, here is an extra row with the corresponding information about f increasing or decreasing.

$x:$	-2	$-\sqrt{\frac{2}{3}}$	0	$\sqrt{\frac{2}{3}}$	1
$f'(x):$	$+$	0	$-$	0	$+$
$f(x):$	$\nearrow$		$\searrow$		$\nearrow$

And this information, the $+, -$, tells us that $f(x)$ is increasing on $\left(-\infty, -\sqrt{\frac{2}{3}}\right)$ and $\left(\sqrt{\frac{2}{3}}, \infty\right)$. And then is decreasing on $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$.

So the function, from left to right, is increasing, decreasing, then increasing again. That means at $x = -\sqrt{\frac{2}{3}}$, since it goes from increasing to decreasing, there is a local maximum. And at $x = \sqrt{\frac{2}{3}}$, since it is going down then up, we have a local minimum. But all we found were the x-values. We should also find the actual point in $f(x)$, so we plug our extrema into the original function. Then $f\left(\sqrt{\frac{2}{3}}\right) = \left(\sqrt{\frac{2}{3}}\right)^3 - 2\sqrt{\frac{2}{3}}$

or approximately, $f(0.816) = -1.089$ is our local minimum y-value. And $f\left(-\sqrt{\frac{2}{3}}\right) = -\left(\sqrt{\frac{2}{3}}\right)^3 + 2\sqrt{\frac{2}{3}}$ or approximately, $f(-0.816) = 1.089$ is our local maximum y-value.

And that is the end of that very long example. But it is the basic trick we will use the rest of the semester. So I suggest trying to repeat this without looking at the example now.

Practice 2:

(1) Repeat the example, without looking.

(2) Find the intervals where $f(x) = 5x^2 - 20x + 3$ is increasing and decreasing. Where are the local extrema if any?

(3) Find the intervals where $f(x) = x^3 - 6x^2 + 2$ is increasing and decreasing. Where are the local extrema if any?

(4) Find the critical points of $g(x) = xe^x$. Where is this increasing and decreasing?

Concavity and inflection points

Now we get to do it all again, but this time thinking about the second derivative. Recall, an **inflection point** is a point where the concavity changes. This is slightly different from a critical point, because not all critical points are mins or maxes. So to keep the analogy for a moment longer, inflection points are more like the local extrema. And we don't have a name for the possible points these might show up, but the process is the same. We will keep the same function from the above example.

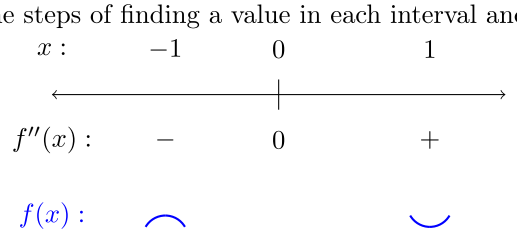
Example W1 continued

Find concavity and inflection points of $f(x)$.

We will first find the points where the concavity *might* change. This is where $f''(x)$ is zero or undefined. (Just like a critical point, but with f'' instead of f'.)

$$f''(x) = 6x$$

Now, we can all see this is a line and so it has no undefined points. And $f''(x) = 0$ when $x = 0$. We will make a number line, even though it is maybe easy enough to see it without one in this example. Here it is in all its glory, skipping the steps of finding a value in each interval and explicitly testing them.



So, on the interval $(-\infty, 0)$, $f(x)$ is concave down and on $(0, \infty)$ $f(x)$ is concave up. Since the concavity changes from down to up at $x = 0$, it is an inflection point.

Practice 3: (1) – (4) Find the concavity and inflection points of the functions from the Practice 2.