MA 121

Section 4.2 Worksheet April, 2020

Directions: It is suggested that you try actually reading the book sections before you watch the videos and attempt to fill out this worksheet. Please read the following problems and explanations carefully. Try to produce full, clear solutions to the exercises. These will help further your understanding.

Section 4.2: We are now looking for the very highest and lowest points on a graph.

This section sees us start the same way as 4.1, we find critical points, $p_1, p_2, \ldots$ of a function on a closed interval $[a, b]$. This is guaranteed to have both a global maximum and minimum by the Extreme value Theorem (EVT).

All we need to do is check the y-value of these points in f AND the end points. So what are $f(p_1), f(p_2), \ldots$ and $f(a)$ and $f(b)$. Then you pick the largest and lowest values. These will be your Global Max and your Global Min.

Practice 1:

(1) Sketch a continuous graph on a closed interval. Find all its critical points, local extrema and global extrema.

(2) Thing of the graph of x^2 on all real numbers. Does this have a global minimum or maximum? Why or why not? What changes if we consider $|x|$ instead of x^2?

(3) For the graph below of f, find all local extrema and global extrema (by estimation) on the following intervals:

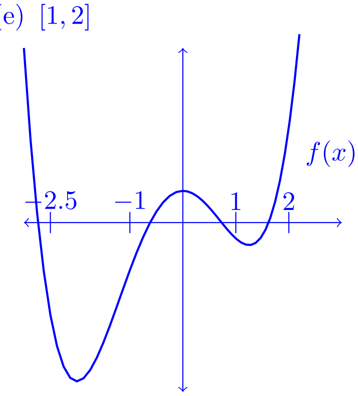
(a) All real numbers $(-\infty, \infty)$.

(b) $[-2.5, 0]$

(c) $[-2.5, 1]$

(d) $[-1, 2]$

(e) $[1, 2]$



Example W1 Let $f(x) = xe^{2x}$. Find the global extrema on $[-3, 1]$.

Solution: Same as last section, we first start by find the critical points. To do this, we will need to find when $f'(x) = 0$ or when f' is undefined, which means using the product rule.

$$f'(x) = [1](e^{2x}) + [2e^{2x}](x) = e^{2x} + 2xe^{2x}$$

Notice, this is all just exponentials and linear functions, so there is nowhere that this is undefined. To find the x where $f'(x) = 0$, we will want to factor. Factoring is now your standard! Please get used to doing it!

$$e^{2x} + 2xe^{2x} = 0$$

$$e^{2x}\left[1 + 2x\right] = 0$$

And we can break this now into two equations, since we are multiplying and getting zero. So $e^{2x} = 0$ and $1 + 2x = 0$. The first one is easy, since e^x is never zero, neither is e^{2x}. So that gives no solutions. But the other, is simply $x = -\frac{1}{2}$.

So the only critical point is $x = -\frac{1}{2}$ and it IS in our domain of $[-3, 1]$, so we keep it. To find the global extrema, we just now plug the critical points and the end points, which in this example are -3 and 1, into f.

$$f(-3) = -3e^{-6} \approx -0.007, \quad f(-\frac{1}{2}) = -\frac{1}{2}e^{-1} \approx -0.184, \quad f(1) = 1e^2 \approx 7.389$$

This is one of the few times that the approximation is often more useful than the exact value in this class. Now, which is the biggest and smallest? Those will be our global max and min!

So there is a global maximum at $x = 1$ and a global minimum at $x = -\frac{1}{2}$. The points on the graph are $(1, e^2)$ and $(-\frac{1}{2}, -\frac{1}{2e})$.

Practice 2:

(1) Repeat the example, without looking.

(2) Find the global extrema of $f(x) = x^4e^x$ on $[-10, 1]$.

(3) Find the points on the graph of the global extrema of $f(x) = x^2 + c$ on $[-1, 1]$, where c is any constant.

Now we will try a slightly different kind of problem. We are going to think about a function on an open interval and find the global extrema.

Example W2 Let $f(x) = \frac{1}{x} + x$. Find the global extrema, if any, on $(0, \infty)$.

Solution: The EVT only gives us global extrema when the interval is closed, so we might not have any in this example!

But let's go through the motions. We will first find $f'(x)$. It is convenient to write $f(x) = x^{-1} + x$.

$$f'(x) = -1x^{-2} + 1 = \frac{-1}{x^2} + 1$$

We need to check when this is undefined and when this is zero. Well right off, we can see that this is undefined at $x = 0$, but this is not in our domain. So we can exclude that. And it is the only place this is undefined. Thus, all we must now do is find when $f'(x) = 0$. Again, it is probably best to factor!

$$\frac{-1}{x^2} + 1 = 0$$
$$\frac{-1}{x^2}[1 - x^2] = 0$$

Here we have two things which we set equal to zero! $\frac{-1}{x^2} = 0$ and $1 - x^2 = 0$. The first has no solutions, so that is done. The second can actually be factored again! This is the difference of squares: $a^2 - b^2 = (a - b)(a + b)$.

$$(1 - x)(1 + x) = 0$$

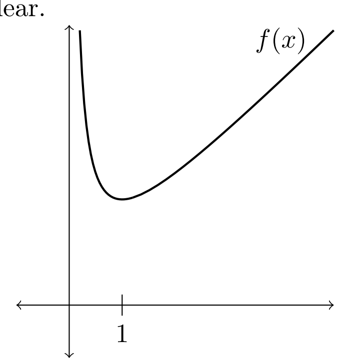
We get a simple equation from each of these, which we solve to get $x = -1$ and $x = 1$.

Since we are only considering positive numbers, we can exclude the negative values. So the only critical point in $(0, \infty)$ is $x = 1$.

What we do now, is see if f is increasing or decreasing around $x = 1$. So, plug in $x = .5$ and $x = 2$ into $f'(x)$. This is like the first derivative test from last section.

$$f'(.5) = \frac{-1}{.5^2} + 1 = -4 + 1 = -3, \qquad f'(2) = \frac{-1}{2^2} + 1 = -.25 + 1 = .75$$

So f changes from decreasing to increasing at $x = 1$. Thus $x = 1$ is a local minimum. But is it a global minimum? Well, yes, it is! Since the function is decreasing on $(0, 1)$ and increasing on $(1, \infty)$, our local minimum must also be the global minimum. Here is a graph of f, which hopefully makes this a little more clear.



Also, notice that f is going up toward ∞ on either side of $x = 1$, so there is no global maximum.

Practice 3:

(1) Repeat Example W2 but on $(-\infty, 0)$.

(2) Find the points on the graph of the global extrema of $f(x) = x^2 + c$ on $(-\infty, \infty)$, where c is any constant.

(3) Find the points on the graph of the global extrema of $f(x) = ax^2 + bx + c$ on $(-\infty, \infty)$, where a, b, and c are constants.