This is just three examples, worked out. Please try the examples **before** you read the solutions!

(1) Find the critical points and inflection points of $f(x)$. Also find the intervals in which f is increasing and decreasing, concave up and concave down. $f(x) = (10 - x^3)^5$

Solution: Since critical points and inflection points are being asked for, we need f' and f''. We will use the chain rule and product rule to get both derivatives. The second to last equation we get by factoring out $30x(10 - x^3)^3$.

$$\begin{aligned}
f'(x) &= 5(10 - x^3)^4[-3x^2] \\
&= -15x^2(10 - x^3)^4 \\
f''(x) &= [-30x]\big((10 - x^3)^4\big) + (-15x^2)\big[4(10 - x^3)^3[-3x^2]\big] \\
&= -30x(10 - x^3)^4 + 180x^4(10 - x^3)^3 \\
&= 30x(10 - x^3)^3\Big[-(10 - x^3) + 6x^3\Big] \\
&= 30x(10 - x^3)^3(-10 + 7x^3)
\end{aligned}$$

To find the critical points, we need to know where the derivative is undefined and zero, but it is always defined. So we set $f'(x) = 0$ and solve. But this immediately gives us two equations: $0 = -15x^2$ and $0 = (10 - x^3)^4$. The first one, gives us $x = 0$ and the second one, we take the fourth root of both sides and get $10 = x^3$, so $x = \sqrt[3]{10} \approx 2.154$. The critical points are $x = 0, \sqrt[3]{10}$.

Now we need to check when the function is increasing or decreasing. It can only possibly change at our critical points, so we need to check a number from each interval $(-\infty, 0)$, $\big(0, \sqrt[3]{10}\big)$, and $\big(\sqrt[3]{10}, \infty\big)$. Not showing the work, it happens that each value is negative. This means the function is decreasing in each interval and never increasing.

Next, we move onto the second derivative. The inflection points are when the concavity changes, so we need to find when $f''(x) = 0$ and what intervals it is positive and negative. So, $0 = 30x(10 - x^3)^3(-10 + 7x^3)$ gives us three equations: $0 = 30x$, $0 = (10 - x^3)^3$, and $0 = -10 + 7x^3$. Solving each of these gives again $x = 0, \sqrt[3]{10}$, but the third gives us $x = \sqrt[3]{\frac{10}{7}} \approx 1.126$. We now need to check the intervals $(-\infty, 0)$, $\left(0, \sqrt[3]{\frac{10}{7}}\right)$, $\left(\sqrt[3]{\frac{10}{7}}, \sqrt[3]{10}\right)$ and $\big(\sqrt[3]{10}, \infty\big)$. Again, not showing the work, the intervals are positive, negative, positive, and negative, respectively. So each value $x = 0, \sqrt[3]{\frac{10}{7}}, \sqrt[3]{10}$ is an inflection point and the intervals $(-\infty, 0) \cup \left(\sqrt[3]{\frac{10}{7}}, \sqrt[3]{10}\right)$ are concave up and $\left(0, \sqrt[3]{\frac{10}{7}}\right) \cup \big(\sqrt[3]{10}, \infty\big)$ are concave down, finishing the problem.

(2) We are making a jar that is a cylinder with a bottom but no top. The material costs 10 cents per square centimeter, for the sides and the bottom costs 2.5 times as much as the sides. If we need the jar to hold 350 cubic centimeters, what is the minimum price to make the jar?

Solution: We are minimizing the cost in this problem. The volume of a cylinder is $V = \pi r^2 h$ and must be 350 cm^3. The surface area comes from a rectangle with vertical side lengths h and around the base side lengths $2\pi r$ and a base of area πr^2, and no top. So $A = 2\pi rh + \pi r^2$, but we want to minimize the cost, not just surface area. So we multiply the rectangle area by 10 and the base area by 25 and remember that this is in cents! $C = 20\pi rh + 25\pi r^2$. Using the constraint equation of $350 = \pi r^2 h$ and solving for either variable will work, but it looks a little easier to eliminate h in the cost equation, so that is what I will solve for: $h = \dfrac{350}{\pi r^2}$. Now our cost equation is $C = 20\pi r \frac{350}{\pi r^2} + 25\pi r^2$ and simplified, $C = \frac{7000}{r} + 25\pi r^2$.

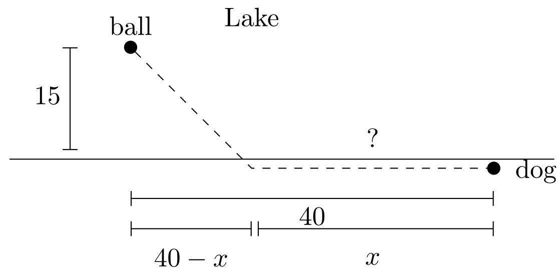
Now we need to find out the domain of r. If $r = 0$, the volume will be zero too, which can't happen, but r can get arbitrarily small, so $0 < r$. And we can make r as big as we want, and h will only get really small, so $r < \infty$. This is okay though, because if r goes to 0, the cost blows up toward infinity, which means it will not be our minimum and if r goes to ∞, the cost again blows up. So our minimum must be some value in $(0, \infty)$.

Next, we find the critical points, C' or $\dfrac{dC}{dr}$, $C' = -7000r^{-2} + 50\pi r$. This is undefined at $r = 0$, but that is not in our domain. So we set it equal to zero and find $7000r^{-2} = 50\pi r$, multiply both sides by r^2 and divide by 50 gives $140/\pi = r^3$ and thus, $r = \sqrt[3]{\dfrac{140}{\pi}} \approx 3.545$. Since we know that C increases to ∞ at both ends of the domain, this must be a minimum.

The problem only asks for the minimum cost, so plugging this back into our equation C gives us $C \approx 2,961.427$.

Answer: The minimum cost to make the jar is about \$29.61.

(3) A dog REALLY wants the ball. And the ball has been thrown into a lake, 40 feet to the left of the dog and 15 feet into the lake. If the dog can run along the shore at 12 feet a second and swim at 4 feet a second, how far should the dog run before it jumps in the water?



Solution: The thing we are optimizing is time, and we want to minimize it. We start by calling the distance the dog runs along the shore x, the rest of the distance is $40 - x$. Then 15 and $40 - x$ form the sides of a right triangle. The hypotenuse of this triangle is then $\sqrt{15^2 + (40-x)^2} = \sqrt{x^2 - 80x + 1825}$. Then we need an equation for the time it takes the dog to get the ball. Recall: velocity = distance/time, so time = distance/velocity. So the time it takes to run along the shore is $x/12$ and the time to swim is $\frac{\sqrt{x^2-80x+1825}}{4}$. Then $T = \frac{x}{12} + \frac{\sqrt{x^2-80x+1825}}{4}$. Note that there is no constraint equation in this problem, but we already only have one variable, so there is no problem.

The domain of x is hopefully pretty easy to see: $0 \leq x \leq 40$. Now we want to find the critical points.

$$\frac{dT}{dx} = \frac{1}{12} + \frac{1}{4}\left[\frac{1}{2}\left(x^2 - 80x + 1825\right)^{\frac{-1}{2}}\right][2x - 80] = \frac{1}{12} + \frac{x - 40}{4\sqrt{x^2 - 80x + 1825}}$$

Here, we see that the denominators are never negative or zero, so this is never undefined. If we set it equal to zero, we get the following:

$$\begin{aligned}
\frac{dT}{dx} &= \frac{1}{12} + \frac{x-40}{4\sqrt{x^2-80x+1825}} \\
0 &= \frac{1}{12} + \frac{x-40}{4\sqrt{x^2-80x+1825}} \\
-\frac{1}{12} &= \frac{x-40}{4\sqrt{x^2-80x+1825}} \\
-1 &= \frac{3(x-40)}{4\sqrt{x^2-80x+1825}} \\
-\sqrt{x^2-80x+1825} &= 3(x-40) \\
\left(x^2-80x+1825\right) &= 9(x-40)^2 \\
x^2-80x+1825 &= 9x^2 - 720x + 14,400
\end{aligned}$$

$$0 = 8x^2 - 640x + 12,575$$

$$x = \frac{640 \pm \sqrt{640^2 - 4(8)(12575)}}{16}$$

$$x = \frac{640 \pm \sqrt{7200}}{16}$$

$$x = \frac{640 \pm 60\sqrt{2}}{16}$$

$$x = 40 \pm \frac{15\sqrt{2}}{4}$$

$$x \approx 34.697, 45.303$$

Since 45.303 is out of the domain, we exclude it. Then we must check the end points, 0 and 40, and the critical point, 34.697.

$T(0) = 10.680$, $T(40) \approx 7.083$, and $T(34.697) \approx 6.869$.

Answer: This shows that the minimum is at $x = 34.697$, so the dog should run 34.697 feet along the shore before it jumps in the water.