Linear Algebra

§2.2: Inverse of a Matrix

Email me.

Idea: How much are Matrices like Numbers?

Num: $+, -, \times, \div$

Matrix: $+, -, \times, ?$ (w/ right size)

$a/b = a \cdot \frac{1}{b} \rightarrow b \cdot \frac{1}{b} = 1 \rightarrow A \cdot B = I$

An *n x n* matrix *A* is called **invertible** if there is an *n x n* matrix *C* such that *AC=I* and *CA=I*. Then *C* is called the **inverse** of *A* and is written A^{-1}.

A matrix which is NOT invertible is said to be **singular** and an invertible matrix is **nonsingular.**

Ex: $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$ $\quad C = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$

$$A\,C = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} =$$

$$= \begin{bmatrix} \underline{4 \cdot -2} + \underline{3 \cdot 3} & 4 \cdot 3 + 3 \cdot -4 \\ 3 \cdot -2 + 2 \cdot 3 & 3 \cdot 3 + 2 \cdot -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C \cdot A = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \cdot 4 + 3 \cdot 3 & -2 \cdot 3 + 3 \cdot 2 \\ \underline{3 \cdot 4} + \underline{-4 \cdot 3} & \underline{3} \cdot 3 + \underline{-4 \cdot 2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow C$ is the inverse of A.

$\Rightarrow \underline{A^{-1} = C}$.

Thm 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The quantity *ad-bc* is called the **determinant** of *A* and we write det *A* = *ad-bc.*

Note: this is only for a 2x2 matrix right now. We will expand this idea later.

End of video 1.

Check that the example fits Thm 4.

Thm 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

(1) $A^{-1}\cdot\vec{b}$ is a sol'n

(2) unique.

Proof ①

$A\vec{x} = \vec{b}$

$A^{-1}\vec{b}$

$$A \cdot (A^{-1}\vec{b}) = (AA^{-1}) \cdot \vec{b}$$
$$= (I_n)\vec{b}$$
$$= \vec{b}.$$

② Assume $\vec{u}$ is also a soln.

$$A\vec{u} = \vec{b}$$
$$A^{-1}(A\vec{u}) = A^{-1}\vec{b}$$
$$\vec{u} = A^{-1}\vec{b}. \quad \blacksquare$$

Try Ex 2, 4!

Thm 6

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

Pf: Matrix inverses are unique:

Assume we have TWO

$A \cdot B = I = BA$ and

$AC = I = CA$

$\Rightarrow$ $$AB = I$$

$$C \cdot (AB) = C \cdot I$$

$$(CA)B = C$$

$$I \cdot B = C$$

$$B = C \quad \checkmark$$

Now, All I have to do, is show $(AB)^{-1}$ and $B^{-1}A^{-1}$ are both inverses of AB.

$$(AB)\cdot(AB)^{-1} = I \quad \checkmark$$

$$(AB)\cdot(B^{-1}A^{-1})$$

$$= A(BB^{-1})A^{-1}$$

$$= A\cdot I\cdot A^{-1}$$

$$= AA^{-1}$$

$$= I$$

$\Rightarrow$ By our lemma that Inverses are Unique,

$$(AB)^{-1} = B^{-1}A^{-1}. \quad \square$$

Thm 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Ex: $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \to \left[A \mid I_n \right] \sim \left[I_n \mid A^{-1} \right]$

$$\begin{bmatrix} 4 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix}$$

$R1 \to R1 - R2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 3 & 2 & 0 & 1 \end{bmatrix}$$

$R2 \to R2 - 3R1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & -1 & -3 & 4 \end{bmatrix}$$

$R2 \to -R2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 3 & -4 \end{bmatrix}$$

$R1 \to R1 - R2$

$$\sim \left[\begin{array}{cc|cc} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} -2 & 3 \\ 3 & -4 \end{matrix}} \end{array}\right]$$

$$[\, I_2 \mid A^{-1} \,]$$

Try Reading Ex 7 in book