Name: ______________________________ MA 253 - Section C

Section 2.2 Worksheet March 30, 2020

Directions: It is suggested that you try actually reading the book sections before you watch the videos and attempt to fill out this worksheet. Please read the following problems and explanations carefully. Try to produce full, clear solutions to the exercises. These will help further your understanding.

Section 2.2: We begin to look into when a matrix has an *inverse.*

For a matrix A, its inverse matrix B, is the matrix such that $AB = BA = I$. This is so special, we will not call the matrix B anymore, we will call it A^{-1}.

Some basic facts:

(1) A must be square, (an $n \times n$ matrix) and then A^{-1} is also the same size square. Then the identity matrix is really I_n, the same size also.

(2) If A^{-1} exists, then it is the unique inverse of A, i.e. the only matrix that is an inverse.

(3) Many matrices, even many square matrices, do not have an inverse.

Example W1:

Let's see what we get when we multiply these two matrices. Recall how we multiply matrices – look back at Section 2.1 for a reminder. $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -\frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} \right] = \left[\begin{bmatrix} 2-1 \\ 2-2 \end{bmatrix} \quad \begin{bmatrix} 1-1 \\ -1+2 \end{bmatrix} \right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

.

Time for you to think:

(1) Check that the example above really is an inverse. That is, find BA and make sure it is I_2.

(2) Use Theorem 4 to check that the A and B from Example W1 are inverses of each other.

(3) What is the determinant of A and of B? For 2×2 matrices, is there a relation between $\det A$ and $\det A^{-1}$?

(4) Let $C = \begin{bmatrix} 3 & 7 \\ 6 & 4 \end{bmatrix}$. Find C^{-1} with Theorem 4.

(5) Why must A be square? If A is a 2×3 matrix, there can be a matrix B such that $AB = I$. What are the sizes of B and I? Find a pair of such matrices. (This can be hard, so use lots of zeros and one's) What can you say about the product BA for the matrices you found?

(6) *Challenge:* Given an invertible matrix A, prove that A^{-1} is unique. (To prove this, pretend that B is another matrix with $AB = BA = I$. Then show B must be equal to A^{-1}.)

Read Theorem 5 very carefully. This means that if we know the inverse of A, we can solve the equation $A\mathbf{x} = \mathbf{b}$ very easily. Recall, we are usually looking for the $\mathbf{x}$, given the $\mathbf{b}$ first. If we already know A^{-1}, then we can just multiply our equation $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1}.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This is important for two reasons. One, it gives us the ability to solve these kinds of equations without much work. And two, it is our first example of *left* multiplication. In the second line, we multiply each side of our equation on the left by A^{-1}. This is now extremely important, because $A^{-1}\mathbf{b} \neq \mathbf{b}A^{-1}$.

Now, read over Example 4 again and verify that it matches with what was just discussed above and Theorem 5.

So, Theorem 4 tells us how to find the inverse of a 2×2 matrix. But what about 3×3? Or bigger? There is an algorithm for this!

First, read Examples 5, 6, and 7. We will use the same matrix E_1 for our first use of the algorithm.

Example W2:

The idea is that we place the matrix we want to find the inverse of next to the identity matrix, $[A\ I]$. Then we row reduce the new big matrix until A is the identity, but we keep track of what happens to I on the right as we do. So eventually you get a matrix that looks like $[I\ B]$. Now, let's try this with E_1! We already know what we want to find for E_1^{-1}, but lets see the algorithm work.

So our matrix will be $[E_1\ I]$ and this is...

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

This matrix we want to make into $[I\ B]$ for some B. So we need to get rid of that -4 in the bottom left corner. So we will use the pivot in the first row first column to do this. We will make row 3 into row 3 + 4 row 1.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4+4\cdot 1 & 0 & 1 & 4\cdot 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 & 1 \end{bmatrix}$$

And that is it! This was an easy one, but you can see we get the inverse we expected on the right half of that matrix. And this is what Theorem 7 tells us too.

Time for you to think:

(1) Check that the example above really is an inverse, so Find $E_1E_1^{-1}$ and $E_1^{-1}E_1$.

(2) Use this algorithm again on the other two matrices from Example 5, E_2 and E_3.

(3) Use the algorithm on $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ and verify that A^{-1} is the same as above.

(4) *Challenge:* Assume that $ad - bc \neq 0$ and use the algorithm on $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and verify Theorem 4.