§2.3 Characterizations of Invertible Matrices

Thm 8

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

TFAE

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. www?
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

Prove?

$a \Leftrightarrow b \Leftrightarrow c \ldots$

$a \Rightarrow b$ 1
$b \Rightarrow a$ 2 unnecessary
$b \Rightarrow c$ 3
$c \Rightarrow b$ 4

$a \xRightarrow{1} b \xRightarrow{3} c \Leftrightarrow d$

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Ex: Is A invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 2 & 3 \\ 9 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & * \\ 0 & -4 & * \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & * \\ 0 & 0 & 3 \end{bmatrix} \leftarrow \neq 0 \leftarrow$$

A has $3 = n$ pivots. By the I.M.T. A is invertible.

$\square$

Def

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{1}$$

$$T(S(\mathbf{x})) = \mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n \tag{2}$$

The next theorem shows that if such an S exists, it is unique and must be a linear transformation. We call S the **inverse** of T and write it as T^{-1}.

Thm 9

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying (1) and (2).

$\Rightarrow$: Assume T is invertible.

There is some $S(\vec{x})$ s.t.

$T(S(\vec{x})) = \vec{x}$ and $S(T(\vec{x})) = \vec{x}$

$T(\vec{x}) = A\vec{x}$

<u>WTS</u>: A is invertible.

We will use (i) of I.M.T.

$\vec{x} \mapsto A\vec{x}$ is onto.

For all $\vec{b} \in \mathbb{R}^n$, there is some $\vec{x}$ s.t. $A\vec{x} = \vec{b}$.

Let $\vec{b} \in \mathbb{R}^n$. Find $\vec{x}$?

Let $\vec{x} = S(\vec{b})$.

$$T(\vec{x}) = A\vec{x}$$
$$\| \qquad \qquad =$$
$$T(S(\vec{b})) = \vec{b}$$

<u>$\Leftarrow$</u>: Assume A is invertible.

A^{-1} exists. $[T(\vec{x}) = A\vec{x}]$

<u>WTS</u>: There is some S s.t.

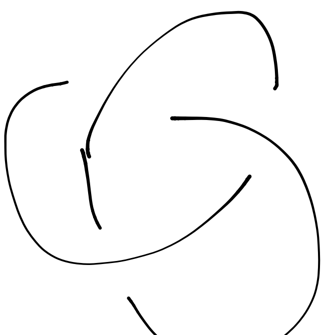
<u>$T(S(\vec{x})) = \vec{x}$</u> and $S(T(\vec{x})) = \vec{x}$.

Let $S(\vec{x}) = A^{-1}\vec{x}$

$$T(S(\vec{x})) = A \cdot S(\vec{x})$$
$$= A \cdot A^{-1} \cdot \vec{x}$$
$$= I_n \vec{x}$$
$$= \vec{x} \quad ✓$$

$$S(T(\vec{x})) = \vec{x} \quad ✓$$

T is invertible iff A is invertible.



Knot!