Math 253 – Week of March 30th

Problems To Chew On

Problem 1. (a) For each real number r, define the linear transformation $D_r : \mathbb{R}^2 \to \mathbb{R}^2$ by setting $D_r(\mathbf{x}) = r\mathbf{x}$.

i. Find the standard matrix for D_r.

ii. Given two real numbers r and s, describe the composition $D_r \circ D_s$.

(b) For each real number θ, define the linear transformation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by setting $R_\theta(\mathbf{x}) = A_\theta \mathbf{x}$, where $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Use matrix multiplication to verify that, for any real numbers θ and ϕ, we have $R_\theta \circ R_\phi = R_{\theta+\phi}$ and explain why this makes sense geometrically.

(c) For each real number a, define the linear transformation $T_a : \mathbb{R}^2 \to \mathbb{R}^2$ by setting $T_a(\mathbf{x}) = M_a \mathbf{x}$, where $M_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.

i. Describe what T_a does to the plane geometrically.

ii. Given two real numbers a and b, use matrix multiplication to find a simpler form for the composition $T_a \circ T_b$. Explain why your answer makes sense geometrically.

Problem 2. A frolic through the land of reflections and rotations.

Consider the linear transformations $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $F_2 : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix representations

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

respectively.

(a) In $\mathbb{R}^2$, plot the triangle with vertices $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and label each vertex.

(b) To understand the linear transformations F_1 and F_2, apply both maps to the vertices $\mathbf{v}_1, \mathbf{v_2}$ and $\mathbf{v}_3$. Please plot your results in the same plane as your triangle from the previous part. You should obtain two triangles. Please label each of the six vertices as $F_k(\mathbf{v}_j)$ for the appropriate values of k and j.

(c) By studying your plot, you should see that transformations F_1 and F_2 are both *reflections*. In other words, F_1 is a reflection over the line ℓ_1 and F_2 is a reflection over the line ℓ_2. What are ℓ_1 and ℓ^2?

(d) You can compose the transformation F_1 with F_2 by first applying the map F_2 to a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and then applying the the map F_1 to the result. This produces a new linear transformation $T = F_1 \circ F_2 : \mathbb{R}^2 \to \mathbb{R}^2$. Thinking about this transformation T diagrammatically, do you expect T to be another reflection? If so, over what line? If not, what is it? Please answer the question by drawing a new plot and following the triangle with vertices $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ by first applying F_2 and then applying F_1 to the result.

(e) Given a vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ which has $\|u\| = \sqrt{u_1^2 + u_2^2} = 1$, called a *unit vector*, let $\mathbf{u}^\perp = \begin{bmatrix} u_2 \\ -u_1 \end{bmatrix}$. Using $\mathbf{u}$ and $\mathbf{u}^\perp$, we define a transformation $\mathrm{Ref}_{\mathbf{u}} : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$\begin{aligned} \mathrm{Ref}_{\mathbf{u}}(\mathbf{x}) &= (\mathbf{u} \cdot \mathbf{x})\mathbf{u} - (\mathbf{u}^\perp \cdot \mathbf{x})\mathbf{u}^\perp \\ &= \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \left(\begin{bmatrix} u_2 \\ -u_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) \begin{bmatrix} u_2 \\ -u_1 \end{bmatrix} \end{aligned}$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\cdot$ denotes the dot product (from Homework 1). To understand this definition, compute the standard matrix representations for the transformations $\mathrm{Ref}_{\mathbf{e}}$ and $\mathrm{Ref}_{\mathbf{d}}$ where $\mathbf{e} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Explain how these transformations relate to F_1 and F_2.

(f) It is not hard to see, for each unit vector $\mathbf{u}$, $\mathrm{Ref}_{\mathbf{u}}$ represents a reflection about a line ℓ in $\mathbb{R}^2$. Based on your response from the previous item, make a conjecture/guess which says how the line ℓ relates to the vector $\mathbf{u}$.

(g) Prove that, for an arbitrary unit vector $\mathbf{u}$, $\mathrm{Ref}_{\mathbf{u}}$ is a linear transformation. Hint: You can use properties of the dot product (which should have appeared in your MA122/MA161-162 text). In working with the definition of $\mathrm{Ref}_{\mathbf{u}}$, it is important to always keep track of which objects are vectors and which objects are scalars.

(h) It is not hard to see that every unit vector can be written in the form $\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ for some angle θ. What does θ represent (geometrically) for the corresponding unit vector?

(i) Writing $\mathbf{u}_\theta = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$, compute the standard matrix representation for $\mathrm{Ref}_{\mathbf{u}_\theta}$.

(j) Based on your observations (much earlier) concerning $T = F_1 \circ F_2$, we expect that $\mathrm{Ref}_{\mathbf{u}_\theta} \circ \mathrm{Ref}_{\mathbf{u}_\phi}$ to be a rotation in the plane. Thus, we should have

$$R_\gamma = \mathrm{Ref}_{\mathbf{u}_\theta} \circ \mathrm{Ref}_{\mathbf{u}_\phi}$$

for some angle γ. Using geometric reasoning (and providing a drawing), what is γ in terms of θ and ϕ?

(k) Justify your answer by computing $\mathrm{Ref}_{\mathbf{u}_\theta} \circ \mathrm{Ref}_{\mathbf{u}_\phi}$ and simplifying.

Problem 3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be its standard matrix representation, which is necessarily an $m \times n$ matrix.

(a) Using the theory we have developed so far in this class, prove the following statement:

If T is both one-to-one and onto, then $m = n$ and so the matrix A is square.

(b) Prove or give a counterexample of the following statement:

If A is a square matrix (i.e., $m = n$), then T is one-to-one.

(c) Prove or give a counterexample of the following statement:

If A is a square matrix (i.e., $m = n$), then T is onto.

Note: By "find a counterexample", we mean to find an explicit square matrix (say a 2×2 or 3×3) for which the conclusion (one-to-one or onto) does not hold. Further, you should demonstrate your assertion. For example, if you can find a 2×2 matrix whose corresponding linear transformation T is not one-to-one, give an example of a non-zero vector $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{0}$.

Problem 4. It is well known that for general matrices A and B, $AB \neq BA$, even when both operations are defined. Here we will discuss what matrices A do have the property, $AB = BA$ for every other matrix B.

(a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the 2×2 matrix with $a, b, c, d \in \mathbb{R}$. Compute AI_2 and I_2A for $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the identity matrix. Does I_2 commute with any matrix A?

(b) Now, we will explore what conditions are required for A to commute with any other B. Let $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Compute AB and BA.

(c) What conditions on a, b, c, d are required for $AB = BA$? Write A simplified with these conditions.

(d) Let $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Compute AC and CA. Again, what conditions will make these two matrices equal? Write this simplified A.

(e) Let $D = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Compute AD and DA.

(f) Write your own theorem based on your observations here.

(g) Make a conjecture on what kinds of $n \times n$ matrices commute with every other $n \times n$ matrix.